Bull. Math. Sci. (2013) 3:205–210
DOI 10.1007/s13373-013-0032-4

ORIGINAL RESEARCH ARTICLE

On supersolvability of fatorized finite groups

Ping Kang · Qingfeng Liu

Received: 18 August 2012 / Revised: 31 January 2013 / Accepted: 6 February 2013 /
Published online: 19 February 2013

Abstract In this paper, we investigate the structure of finite groups that are products of two supersolvable groups and gain a sufficient condition for a group to be supersolvable. Our main theorem is the following: Let the group $G = HK$ be the product of the subgroups H and K. Assume that H permutes with every maximal subgroup of K and K permutes with every maximal subgroup of H. If H is supersolvable, and K is nilpotent and K is δ-permutable in H, where δ is a complete set of Sylow subgroups of H, then G is supersolvable. Some known results are generalized.

Keywords Supersolvable groups · δ-Permutable subgroups · Finite groups

Mathematics Subject Classification (2010) 20D10 · 20D20

1 Introduction

All groups considered in this paper are finite. we use conventional notions and notation, as in [1].

Communicated by S. K. Jain.

The research of the authors is supported by the National Natural Science Foundation of China (10771132), the Research Grant of Tianjin Polytechnic University and Shanghai Leading Academic Discipline Project (J50101).

P. Kang (✉)
Department of Mathematics, Tianjin Polytechnic University,
Tianjin 300387, People's Republic of China
e-mail: kangping2929@163.com

Q. Liu
Department of Basic Sciences, Shandong Water Polytechnic,
Rizhao 276826, People's Republic of China

It is well known that a group which is the product of two supersolvable groups is not necessarily supersolvable, even if the two factors are normal subgroups of the group. The first result we know in this direction is due to Baer. He proved in [2] that if a group G is the product of two normal supersolvable subgroups and G' is nilpotent, then G is supersolvable. Friesen [3] proved that if a group G is the product of two normal supersolvable subgroups of coprime indices, then G is also supersolvable. The search for generalisations of Baer's and Friesen's results has been a fruitful topic of investigation recently (see [4–6]).

Most of the generalisations centre around replacing normality of the factors by different permutability conditions. Asaad and Shaalan proved in [7, Theorem 3.2] the following generalisations of Baer's and Friesen's results:

Theorem A *Suppose that H is nilpotent subgroup of G, K is supersolvable subgroup of G and that $G = HK$. Suppose further that H permutes with every subgroup of K and K permutes with every subgroup of H, then G is supersolvable.*

Later this result was improved in [8]. More precisely, they proved:

Theorem B *Let the group $G = HK$ be the product of the subgroups H and K. Assume that H permutes with every maximal subgroup of K and K permutes with every maximal subgroup of H. If H is supersolvable, and K is nilpotent and K permutes with every Sylow subgroup of H, then G is supersolvable.*

The aim of this paper is to take these studies further and generalise the above mentioned results.

At first we introducing the following definition. In 2003, Asaad and Heliel in [9] introduced a new embedding property, namely the δ-permutability, of subgroups of a group. δ is called a complete set of Sylow subgroups of G if for each prime $p \in \pi(G)$ (the set of distinct primes dividing $|G|$), δ contains exactly one Sylow p-subgroup of G, G_p say. A subgroup H of a finite group G is said to be δ-permutable in G if H permutes with every member of δ. We will follow [10] and consider δ-permutability in some subgroup of a group G and the following definition is definition 1.1 in [10]:

Definition If K and H are subgroups of a group G, and δ is a complete set of Sylow subgroups of H, then K is said to be δ -permutable in H if K permutes with every member of δ.

It is clear that if K permutes with every Sylow subgroup of H, then K is δ-permutable in H. But the converse does not hold in general.

In this paper, we mainly study the structure of finite groups which are products of one supersolvable subgroup and one nilpotent subgroup, and obtain a sufficient condition to guarantee its supersolvability. Our main result is the following:

Theorem C *Let the group $G = HK$ be the product of the subgroups H and K. Assume that H permutes with every maximal subgroup of K and K permutes with every maximal subgroup of H. If H is supersolvable, and K is nilpotent and K is δ-permutable in H, where δ is a complete set of Sylow subgroups of H, then G is supersolvable.*

2 Preliminary results

In this section, we give some results that are needed in this paper.

Lemma 2.1 ([8]) *Let the group $G = HK$ be the product of the subgroups H and K. If H is supersolvable, K is nilpotent and K permutes with every maximal subgroup of H, then G is solvable.*

Let δ be a complete set of Sylow subgroups of G. If $N \trianglelefteq G$, we shall denote by δN the following set of subgroups of G:

$$\delta N = \left\{G_p N : G_p \in \delta\right\},$$

and by $\delta N/N$ the following set of subgroups of G/N:

$$\delta N/N = \left\{G_p N/N : G_p \in \delta\right\},$$

and by $\delta \bigcap N$ the following set of subgroups of G:

$$\delta \bigcap N = \left\{G_p \bigcap N : G_p \in \delta\right\}$$

Lemma 2.2 ([9]) *Let δ be a complete set of Sylow subgroups of G, U be a δ -permutable subgroup of G, and N a normal subgroup of G. Then:*

(i) *$\delta \bigcap N$ and $\delta N/N$ are complete sets of Sylow subgroups of N and G/N, respectively.*
(ii) *UN/N is a $\delta N/N$ -permutable subgroup of G/N.*
(iii) *If $U \leq N$, then U is a $\delta \bigcap N$ -permutable subgroup of N.*

3 Proof of Theorem C

Proof Assume that the result is not true and let G be a counterexample of minimal order. By Lemma 2.1, we know that G is solvable. If $1 \neq N$ is a normal subgroup of G, then G/N is the product of the subgroups HN/N and KN/N. Moreover, HN/N permutes with every maximal subgroup of KN/N and KN/N permutes with every maximal subgroup of HN/N, HN/N is supersolvable and KN/N is nilpotent. By Lemma 2.2 it is clear that KN/N is a $\delta N/N$ -permutable subgroup of HN/N. The minimal choice of G implies that G/N is supersolvable. Consequently, G has a unique minimal normal subgroup N which is abelian and complemented in G by a core-free maximal subgroup M of G. Let p be the prime dividing $|N|$ and let q be the largest prime dividing $|G|$.

Suppose that $p \neq q$. Let H_q be a Sylow q-subgroup of H. Then H_q is a normal subgroup of H because H is supersolvable. Thus $H_q \in \delta$. Moreover K has a unique Sylow q-subgroup K_q because K is nilpotent. Applying [11, 1.3.1 and 1.3.3], we have that H_q permutes with K_q^g for each $g \in G$. Since $O_q(G) = 1$, it follows that $[H_q^G, K_q^G] = 1$ by [11, 2.5.1]. It is quite clear that we can assume that either $H_q^G \neq 1$ or $K_q^G \neq 1$ because otherwise G would be a q'-group.

At first we assume that $H_q^G \neq 1$. Then we have that N is contained in H_q^G. Therefore $[N, K_q^G] = 1$ and $K_q^G \leq C_G(N) = N$. Hence $K_q^G = 1$ and K is a q'-group. Since every Sylow q-subgroup of M is a Sylow q-subgroup of G, we may assume that H_q is contained in M. As M is supersolvable, it follows that H_q is normalized by M. If $G = N_G(H_q)$, then N is contained in H_q, a contradiction. So $M = N_G(H_q)$. This implies that H is contained in M.

By the fact that K permutes with every maximal subgroup of H, it follows that $H \bigcap K$ permutes with every maximal subgroup of H. Moreover, K is nilpotent, so every maximal subgroup of K is a normal subgroup of K. Hence $H \bigcap K$ permutes with every maximal subgroup of K. Since $H \bigcap K$ is nilpotent, it follows that $H \bigcap K$ is contained in $F(H)$ by [12, Satz 2.10(b)]. In particular, $H \bigcap K$ is subnormal in H. Since $H \bigcap K$ is also subnormal in K, it follows that $H \bigcap K$ is subnormal in G by [11, 7.7.1]. Thus $H \bigcap K$ is a nilpotent subnormal subgroup of G. Hence $H \bigcap K$ is contained in N. This means that $H \bigcap K = 1$ because N is complemented by M. By [13, Proposition 3.3(c)] , every maximal subgroup of H permutes with every maximal subgroup of K.

Let M_1 and M_2 be maximal subgroup of H and K respectively. If $g \in G$, then $g = hk$, where $h \in H$ and $k \in K$. Since M_1^h is a maximal subgroup of H, it follows that M_1^h permutes with M_2. The normality of M_2 in K implies that M_1^{hk} permutes with M_2. Therefore $M_1^g M_2$ is a subgroup of G for every $g \in G$. By [11, 2.5.1], it follows that $[M_1, M_2]$ is a subnormal subgroup of G. By [11, 7.5.5], we know that N normalizes $[M_1, M_2]$.

On the other hand, $M = H(M \bigcap K)$. Hence $M \bigcap K$ is a maximal subgroup of K. Therefore if H_0 is a maximal subgroup of H, we have that $[H_0, M \bigcap K]$ is a normal subgroup of $H_0(M \bigcap K)N$ such that $[H_0, M \bigcap K] \bigcap N \leq H_0(M \bigcap K) \bigcap N = 1$. This means that $M \bigcap K$ centralizes H_0. Hence $M \bigcap K$ centralizes every maximal subgroup of H. Suppose that H has a unique maximal subgroup. Then H is a cyclic q-group. Therefore K is a Hall q'-subgroup of G and then N is contained in K. Since K is nilpotent, it follows that $K = N(M \bigcap K)$ is a p-group. Consequently $|K : M \bigcap K| = |N| = p$ and N is cyclic, a contradiction. Hence we may assume that H has at least two maximal subgroups. Then $M \bigcap K$ centralizes H and so $M \bigcap K$ is a normal subgroup of G. Since N is the unique minimal normal subgroup of G, it follows that $M \bigcap K = 1$. This implies that K is a cyclic group of order p (notice that p must divide $|K|$ because otherwise $N \leq H \leq M$, a contradiction). On the other hand, $M = H(M \bigcap K)$ and $G = HK = NM = NH$. Then there exists an element $h \in H$ and Sylow p-subgroups G_p and H_p of G and H respectively such that $G_p = H_pK = NH_p^h$. By order considerations, it follows that $|K| = |N| = p$ and then G is supersolvable, a contradiction.

Now we suppose that $K_q^G \neq 1$. Arguing as above, we have that H is a q'-group, $H \bigcap K = 1$ and K is contained in M. Moreover $M \bigcap H$ is a maximal subgroup of H. With similar arguments to those used above, it follows that $M \bigcap H$ centralizes every maximal subgroup of K. If K has only a unique maximal subgroup, then K is a cyclic q-group and then N is contained in H. If not, $M \bigcap H$ centralizes K and so $M = (M \bigcap H) \times K$. Since K is nilpotent and $O_p(K)$ is contained in $O_p(M) = 1$, it follows that K is a p'-group. Consequently N is contained in H. Hence, in both cases, we have that N is contained in H. Therefore $H = N(M \bigcap H)$. Since $M \bigcap H$ is a maximal subgroup of H and H is supersolvable, it follows that $|H : M \bigcap H|$ is a prime number by [14, VII, 2.2]. Consequently $|N| = |H : M \bigcap H| = p$ and G is supersolvable, a contradiction.

Assume now that p is the largest prime dividing $|G|$. Since M is supersolvable and $O_p(M) = 1$, it follows M is a p'-group and so N is a Sylow p-subgroup of G. Let $K_{p'}$ be a Hall p'-subgroup of K. Then $K_{p'}$ is normal in K. Suppose that $(H \bigcap K)K_{p'}$ is a proper subgroup of K and let K_0 be a maximal subgroup of K containing $(H \bigcap K)K_{p'}$. Then HK_0 is a proper subgroup of G. Denote $S = HK_0$. If $\mathrm{Core}_G(S) = 1$, then $S \bigcap N = 1$ and $G = SN$. Hence $|N| = |G : S| = |HK : HK_0| = |K : K_0| = p$. This implies that G is supersolvable, a contradiction. Assume that $\mathrm{Core}_G(S) \neq 1$. Then N is contained in HK_0 and so N is a Sylow p-subgroup of HK_0. Therefore $|N| = |H|_p|K_0|_p/|H \bigcap K_0|_p = |H|_p|K|_p/|H \bigcap K|_p$. Since $H \bigcap K = H \bigcap K_0$, it follows that $|K|_p = |K_0|_p$. In particular, K_0 contains a Sylow p-subgroup of K. Hence $K = K_0$, a contradiction. Consequently $K = (H \bigcap K)K_{p'}$. This implies that $H \bigcap K$ is a Sylow p-subgroup of K (notice that $H \bigcap K \leq F(G) = N$) and then H contains N. Thus $H = N(H \bigcap M)$.

By hypothesis, K is δ-permutable in H, where δ is a complete set of Sylow subgroups of H. Hence we can easily deduce that K permutes with every Hall p'-subgroup of H. Let $H_{p'}$ be one of them. Then $KH_{p'} = (H \bigcap K)K_{p'}H_{p'}$, where $K_{p'}$ is the unique Hall p'-subgroup of K, is a subgroup of G. Notice that in this case $K_{p'}H_{p'}$ is a Hall p'-subgroup of $KH_{p'}$. Hence we have proved that $K_{p'}$ permutes with every Hall p'-subgroup of H. In fact, for every $g \in G$, $K_{p'}$ permutes with $H_{p'}^g$. Since $O_{p'}(G) = 1$, it follows that $[H_{p'}^G, K_{p'}^G] = 1$. If $K_{p'} = 1$, then $K = H \bigcap K \leq H$, a contradiction. Hence $K_{p'}^G \neq 1$ and so $H_{p'} \leq C_G(N) = N$. This implies that $H = N$ and K is a core-free maximal subgroup of G complementing H. Let N_0 be a maximal subgroup of N. Then N_0K is a subgroup of G. Since K is maximal in G, it follows that either $G = N_0K$ or $K = N_0K$. If $G = N_0K$, then $N = N_0$, a contradiction. Hence $K = N_0K$ and $N_0 = 1$. Consequently, N is cyclic and G is supersolvable, final contradiction. □

Remark 3.1 It is not true in general that a product of a supersolvable group and a nilpotent group is supersolvable as $PSL(2, 7)$ shows. In addition, permutability of H with every maximal subgroup of K is essential in order to get supersolvability in Theorem C. For example, let G be the symmetric group of degree four, then $G = HK$ where K is the 4-klein group and H is isomorphic to the symmetric group of degree 3. Since K is normal in G, we have that K permutes with every subgroup of H and H is supersolvable. However, G is not supersolvable. Notice that H does not permute with the maximal subgroups of K.

Acknowledgments The authors are very grateful to the referee for his/her valuable suggestions. It should be said that we could not have polished the final version of this paper well without their outstanding efforts.

References

1. Robinson, D.J.S.: A Course in the Theory of Groups. Springer, New york (1980)
2. Baer, R.: Classes of finite groups and their properties. Ill. J. Math. **1**, 115–187 (1957)
3. Friesen, D.R.: Products of normal supersolvable subgroups. Proc. Am. Math. Soc. **30**, 46–48 (1971)
4. Ballester-Bolinches, A., Pérez Ramos, M.D., Pedraza-Aguilera, M.C.: Totally and mutually permutable products of finite groups. In: Groups St. Andrews 1997 in Bath I. London Mathematical Society. Lecture Note Series, vol. 160, pp. 65–68. Cambridge University Press, Cambridge (1999)
5. Carocca, A.: p-supersolvability of factorized finite groups. Hokkaido Math. J. **21**, 395–403 (1992)
6. Carocca, A., Maier, R.: Theorems of Kegel-Wielandt type. In: Groups St. Andrews 1997 in Bath I. London Mathematical Society. Lecture Note Series, vol.160, pp.195–201. Cambridge University Press, Cambridge (1999)
7. Asaad, M., Shaalan, A.: On the supersolvability of finite groups. Arch. Math. **53**, 318–326 (1989)
8. Ballester-Bolinches, A., Cossey, J., Pedraza-Aguilera, M.C.: On products of finite supersolvable groups. Commun. Algebra **29**(7), 3145–3152 (2001)
9. Asaad, M., Heliel, A.A.: On permutable subgroups of finite groups. Arch. Math. **80**, 113–118 (2003)
10. Kong, Q., Guo, X.: On δ-permutability of maximal subgroups of Sylow subgroups of finite groups. Arch. Math. **91**, 106–110 (2008)
11. Amberg, B., Franciosi, S., de Giovanni, F.: Products of Groups. Clarendon, Oxford (1992)
12. Maier, R.: Zur vertauschbarkeit und subnormalit$\ddot{a}$t von untergruppen. Arch. Math. **53**, 110–120 (1989)
13. Carocca, A.: Products of mutually f-permutable subgroups in a finite group (preprint)
14. Doerk, K., Hawkes, T.: Finite Solvable Groups. Walter De Gruyter, Berlin (1992)